Topic 7.1

Modeling Situations with Differential Equations

Differential Equations in Context

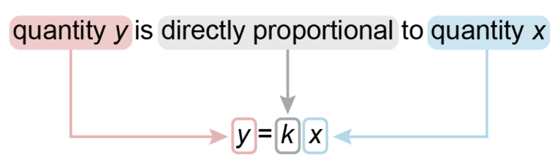
A **differential equation** is an equation that relates one or more functions of an independent variable with their derivatives. For example, the resulting equation after differentiation in a related rates context (Topic 4.4) is a differential equation.

A context can be modeled with a differential equation if it relates quantities to their **rates of change**. Some examples are given in the following table.

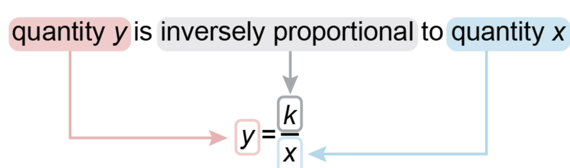
Context	Differential equation
A particle moves along the *x*-axis such that the rate of change in its position is equal to 3 times its current position.	$\frac{dx}{dt} = 3x$
The rate of change in the area of an equilateral triangle is equal to half the product of its base length and the rate of change in height.	$\frac{dA}{dt} = \frac{1}{2} b \frac{dh}{dt}$
The rate of change in the perimeter of a rectangle is equal to 2 times the sum of the rates of change in the length and width.	$\frac{dP}{dt} = 2\left(\frac{dl}{dt} + \frac{dw}{dt}\right)$

Some contexts may relate these quantities and derivatives with a proportional relationship.

If a quantity *y* is **proportional** or **directly proportional** to another quantity *x*, then there is some constant *k* such that $\boldsymbol{y = kx}$.



If the quantity *y* is **inversely proportional** to the quantity *x*, then there is a constant *k* such that $y = \frac{k}{x}$.



Example

A particle moves in the *xy*-plane in such a way that the rate of change in its vertical position *y* with respect to time *t* is **directly proportional** to the rate of change in its horizontal position *x* with respect to *t*.

To model this context with a differential equation, first translate the rate of change in the vertical position *y* with respect to *t* as $\frac{dy}{dt}$ and the rate of change in the horizontal position *x* with respect to *t* as $\frac{dx}{dt}$. Then multiply by a constant of proportionality *k*.

$$\frac{dy}{dt} = k\frac{dx}{dt}$$

Constant *k* could be any real number except 0, so $\frac{dy}{dt} = \frac{dx}{dt}$ (where *k* = 1), $\frac{dy}{dt} = -\pi\frac{dx}{dt}$, and $\frac{dy}{dt} = \frac{\sqrt{3}}{14}\frac{dx}{dt}$ are all examples of equations that **could** model the given context.

7.1 Check for Understanding

1. **A frog population *F* grows with respect to time *t* at a rate inversely proportional to the square of the population. Which of the following is a differential equation that could describe this relationship?**

   A. $\frac{dF}{dt} = \frac{100}{F}$

   B. $\frac{dF}{dt} = \frac{100}{F^2}$

   C. $\frac{dF}{dt} = 100F$

   D. $\frac{dF}{dt} = 100F^2$

2. **A container with a square base with side length *s* is filling with water. The volume *V*(*t*) of water in the container grows with respect to time *t* at a rate proportional to the area of the base. Which of the following differential equations could model the rate of change in volume?**

   A. $\frac{dV}{dt} = 3t$

   B. $\frac{dV}{dt} = 3V$

   C. $\frac{dV}{dt} = 3s$

   D. $\frac{dV}{dt} = 3s^2$