Topic 3.1

The Chain Rule

The Chain Rule

A **composite function** is a function of the form $f(g(x))$, or $f(u)$, where f is the outside function and $u = g(x)$ is the inside function. The **chain rule** is a method to find the derivative of a composite function.

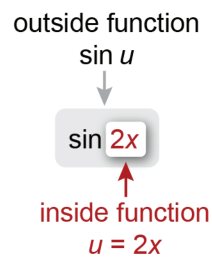
The derivative of a function of the form $f(g(x))$ is the derivative of the outside function with the inside function as input $f'(g(x))$ multiplied by the derivative of the inside function $g'(x)$.

chain rule

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

Example

Consider the function $f(x) = \sin 2x$. The function f is a composite function with the outside function $\sin x$ and the inside function $2x$.



To differentiate the composite function f, differentiate the outside function $\sin x$, plug in the inside function $2x$, and multiply by the derivative of the inside function $2x$.

$f(x) = \sin 2x$	Given function
$f'(x) = \frac{d}{dx}[\sin 2x]$	Differentiate both sides
$f'(x) = \cos 2x \cdot \frac{d}{dx}[2x]$	Apply sine rule to outside function and leave inside function intact
$f'(x) = \cos 2x \cdot 2$	Apply power rule to differentiate inside function
$f'(x) = 2\cos 2x$	Simplify

Therefore, the derivative of $f(x) = \sin 2x$ is $f'(x) =$ **$2\cos 2x$**.

To see another representation of the chain rule, let y be a composite function of x, where $u(x)$ is the inside function. The derivative of y with respect to x is $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

The first term of the derivative $\frac{dy}{du}$ is the **derivative of *y* with respect to *u***, which means the derivative of the outside function *y* with the inside function *u* left intact. The second term $\frac{du}{dx}$ is the **derivative of the inside function *u* with respect to *x***.

chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

The derivative rules given in Unit 2 can be combined with the chain rule.

Let $u = f(x)$ be a function and consider each of the functions examined in Unit 2 as composite functions with *u* as the inside function. The derivatives are the given derivative functions multiplied by the derivative of *u*.

Power	$\frac{d}{dx}[u^n] = nu^{n-1} \cdot u'$
Natural exponential	$\frac{d}{dx}[e^u] = e^u \cdot u'$
Exponential	$\frac{d}{dx}[a^u] = (\ln a)a^u \cdot u'$
Natural log	$\frac{d}{dx}[\ln(u)] = \frac{1}{u} \cdot u'$
Sine	$\frac{d}{dx}[\sin u] = \cos u \cdot u'$
Cosine	$\frac{d}{dx}[\cos u] = -\sin u \cdot u'$
Tangent	$\frac{d}{dx}[\tan u] = \sec^2 u \cdot u'$
Secant	$\frac{d}{dx}[\sec u] = \sec u \tan u \cdot u'$
Cosecant	$\frac{d}{dx}[\csc u] = -\csc u \cot u \cdot u'$
Cotangent	$\frac{d}{dx}[\cot u] = -\csc^2 u \cdot u'$

3.1 Check for Understanding

1. $\frac{d}{dx}[\sin(\cos x)] =$

A. cos(cos *x*)

B. cos(−sin *x*)

C. sin(−sin *x*) · cos *x*

D. cos(cos *x*) · (−sin *x*)

x	*f*	*g*	*f′*	*g′*
1	0	3	5	1
2	6	1	2	3
3	1	2	8	7

2. The table above gives selected values of two differentiable functions *f* and *g* along with their derivatives. If *y* = *f*(*g*(*x*)), what is the value of $\frac{dy}{dx}$ at *x* = 2 ?

A. 3

B. 6

C. 8

D. 15