Topic 2.1

Defining Average and Instantaneous Rates of Change at a Point

Difference Quotient

Limits can be used to formally define the instantaneous rate of change (IROC) of a function at a point, which was defined in Topic 1.1 as the **limit of the average rate of change (AROC)** over an interval as the **endpoints approach the same value**.

As discussed in Topic 1.1, the AROC of a function *f* between two points (a, $f(a)$) and (b, $f(b)$) is given by the **slope of the secant line** between the two points.

average rate of change over [*a*, *b*]

$$\frac{f(b) - f(a)}{b - a}$$

To construct a limit of the AROC as the endpoints approach the same value, one of the endpoints must be a variable, so substitute x for b to get an expression known as the **difference quotient**, $\frac{f(x)-f(a)}{x-a}$.

Let h denote the distance along the x-axis between a and x, so $h = x - a$ or, when solved for x, $x = a + h$. Substitute the expression $a + h$ for x in the difference quotient to find another form of the difference quotient.

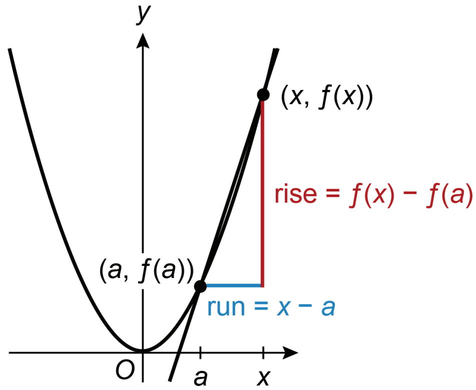
$\frac{f(x) - f(a)}{x - a}$	AROC formula
$\frac{f(a + h) - f(a)}{a + h - a}$	Substitute $x = a + h$
$\frac{f(a + h) - f(a)}{h}$	Subtract in denominator

Therefore, $\frac{f(x)-f(a)}{x-a}$ and $\frac{f(a+h)-f(a)}{h}$ are both functions that give the AROC of *f* between (a, $f(a)$) and any point (x, $f(x)$) in the domain of *f*.

Example 1

The graph below shows the function $f(x) = x^2$ for some interval that includes a.

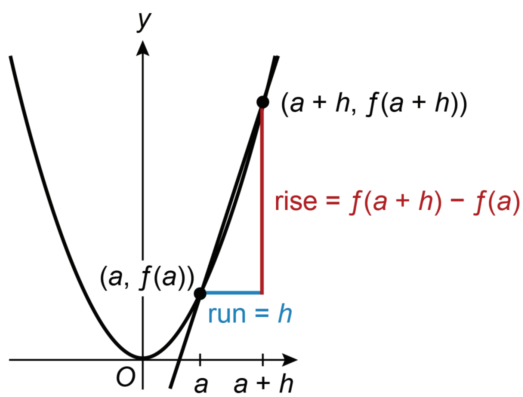
The **AROC** between (a, $f(a)$) and any point (x, $f(x)$) on the graph of the function is given by $\frac{f(x)-f(a)}{x-a}$, the **slope of the secant line** running through (a, $f(a)$) and the generic point (x, $f(x)$).



$$\text{Slope} = \frac{f(x) - f(a)}{x - a}$$

Example 2

Let h be defined as the distance between a and the generic value x. Therefore, $x = a + h$ and the **AROC** between $f(a)$ and $f(x)$ is given by $\frac{f(a+h)-f(a)}{h}$.



$$\text{Slope} = \frac{f(a+h) - f(a)}{h}$$

As defined previously, the **IROC** of a function f at $x = a$ is given by the limit of the AROC over an interval that contains a as the endpoints approach a.

The IROC is also known as the **derivative**, and it is the focus of the next four units of this course.

2.1 Check for Understanding

x	0	1	2	3	4	5
$f(x)$	−3	2	10	7	13	12

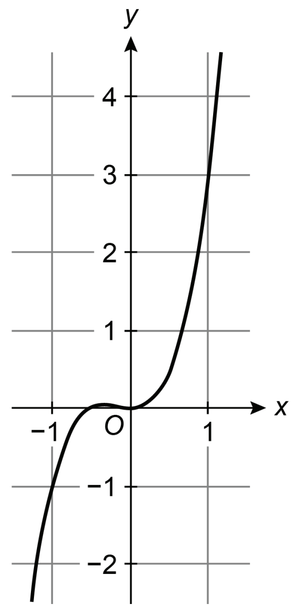
1. The table above gives selected values of a function *f*. Which of the following is the average rate of change on [0, 4] ?

A. 3

B. 4

C. 5

D. 16



Graph of f

2. The graph of a function *f* is given above. Which of the following is the slope of the secant line from $x = -1$ to $x = 1$?

A. 1

B. 2

C. 3

D. 4