Topic 10.1

Defining Convergent and Divergent Infinite Series

Infinite Series and Partial Sums

A **summation** is the sum of objects that follow a particular pattern given in **summation notation**, denoted by a sigma $\sum$.

The equation below the sigma defines the integer **starting value** of a variable n, and the value above the sigma is the **ending value**. To calculate the sum, substitute each integer from the starting value to the ending value into the expression to the right of the sigma and add the results.

$$\sum_{n=1}^{4} n^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30$$

+1 +1 +1

The example summation has a finite ending value, but some summations instead **approach infinity**. Such a summation is called an **infinite series**, or just a **series**.

Replacing the ending value 4 in the summation with infinity gives an infinite series that represents the sum of the square of every integer from $n = 1$ to infinity. This particular sum grows **larger and larger unbounded**, so it is said to **diverge**.

$$\sum_{n=1}^{\infty} n^2 = 1^2 + 2^2 + 3^2 + \cdots = \infty$$

Now consider the series that adds the reciprocals of powers of 2: $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$.

First consider the sum of the series with only one term. Then include more and more terms to observe how the sum changes with increasing numbers of terms.

1 term $\quad \sum_{n=1}^{1} \frac{1}{2^n} = \frac{1}{2} = 0.5$

2 terms $\quad \sum_{n=1}^{2} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} = 0.75$

3 terms $\quad \sum_{n=1}^{3} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 0.875$

4 terms $\quad \sum_{n=1}^{4} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = 0.9375$

$\vdots$

The sum continues to grow as n increases, but the growth is **bounded**. Each term added increases the sum, which becomes closer to but never exceeds 1.

Therefore, the series **converges** to 1.

$$\text{infinite terms} \quad \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1$$

Partial Sums and the Sum of an Infinite Series

A **partial sum** of an infinite series is the finite sum of the first few terms up to a finite ending value. For example, the sum $\sum_{n=1}^{4} n^2$ is the sum of the first four terms of the infinite series $\sum_{n=1}^{\infty} n^2$, so it is called the **4th partial sum** and denoted $\boldsymbol{S_4}$.

In general, the ***k*th partial sum $\boldsymbol{S_k}$** is the sum of the first k terms of an infinite series. Let a_n be the **general term**, or nth term for any integer n, in a series. The kth partial sum is given by:

$$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + \cdots + a_k$$

As k increases, the partial sum S_k adds more terms. If the infinite series **converges** as k approaches infinity, the partial sum S_k approaches the **sum of the infinite series $\boldsymbol{S}$**.

$$S = \lim_{k\to\infty} S_k = \lim_{k\to\infty} \sum_{n=1}^{k} a_n$$

Therefore, convergence or divergence of an infinite series, as well as the value of its sum if it converges, can sometimes be determined by examining the partial sums as the number of terms increases toward infinity.

For example, to show mathematically that $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$, rewrite the partial sums in a form that follows a pattern. Examine the partial sums to find a **general expression** that gives the value of the sum for any number of terms k, and then calculate the limit as k approaches infinity.

$$\text{1 term} \quad \sum_{n=1}^{1} \frac{1}{2^n} = \frac{1}{2} = 1 - \frac{1}{2} \rightarrow 1 - \frac{1}{2^1}$$

$$\text{2 terms} \quad \sum_{n=1}^{2} \frac{1}{2^n} = \left(1 - \frac{1}{2}\right) + \frac{1}{4} = 1 - \frac{1}{4} \rightarrow 1 - \frac{1}{2^2}$$

$$\text{3 terms} \quad \sum_{n=1}^{3} \frac{1}{2^n} = \left(1 - \frac{1}{4}\right) + \frac{1}{8} = 1 - \frac{1}{8} \rightarrow 1 - \frac{1}{2^3}$$

$$\text{4 terms} \quad \sum_{n=1}^{4} \frac{1}{2^n} = \left(1 - \frac{1}{8}\right) + \frac{1}{16} = 1 - \frac{1}{16} \rightarrow 1 - \frac{1}{2^4}$$

$$\vdots$$

$$k \text{ terms} \quad \sum_{n=1}^{k} \frac{1}{2^n} = 1 - \frac{1}{2^k}$$

Therefore, the *k*th partial sum is $S_k = 1 - \frac{1}{2^k}$. Calculate the limit as *k* approaches infinity to verify that the sum of the infinite series is 1.

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{k \to \infty} S_k$$ Sum of series equals infinite limit of partial sums

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{k \to \infty} \left(1 - \frac{1}{2^k}\right)$$ Substitute $S_k = 1 - \frac{1}{2^k}$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1 - 0$$ Evaluate limit

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$ Subtract

10.1 Check for Understanding

1. **The series $\sum_{n=1}^{\infty} 1$ converges.**

   True

   False

2. $\sum_{n=1}^{\infty} \frac{3}{10^n} =$

   A. 0

   B. $\frac{3}{10}$

   C. $\frac{1}{3}$

   D. The series diverges.