Calculus BC Formulas

Section 1: Limits

Properties of limits as $x \to c$

If L, M, c, and k are real numbers and $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, then the following properties are true:

Limit of a constant	$\lim_{x\to c} k = k$
Limit of x	$\lim_{x\to c} x = c$
Sum rule	$\lim_{x\to c} (f(x) + g(x)) = L + M$
Difference rule	$\lim_{x\to c} (f(x) - g(x)) = L - M$
Product rule	$\lim_{x\to c} (f(x) \cdot g(x)) = L \cdot M$
Constant multiple rule	$\lim_{x\to c} (k(f(x)) = k \cdot L$
Quotient rule	$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M},\ M \neq 0$
Power rule	$\lim_{x\to c} (f(x))^{r/s} = L^{r/s}$, if r and s are integers, and $s \neq 0$
Limit of a composite function	$\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right)$, if f is a continuous function

Properties of limits as $x \to \pm\infty$

If L, M, c, and k are real numbers and $\lim_{x\to\pm\infty} f(x) = L$ and $\lim_{x\to\pm\infty} g(x) = M$, then the following properties are true:

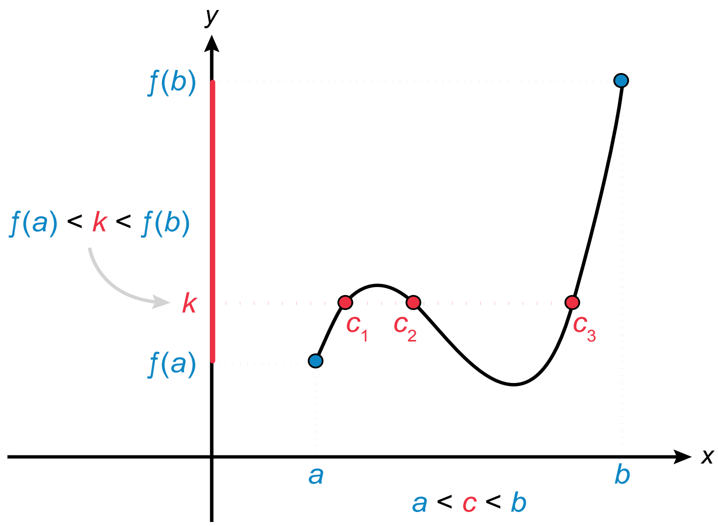
Constant rule	$\lim_{x\to\pm\infty} c = c$
Sum rule	$\lim_{x\to\pm\infty} (f(x) + g(x)) = L + M$
Difference rule	$\lim_{x\to\pm\infty} (f(x) - g(x)) = L - M$
Product rule	$\lim_{x\to\pm\infty} (f(x) \cdot g(x)) = L \cdot M$
Constant multiple rule	$\lim_{x\to\pm\infty} (k(f(x)) = k \cdot L$
Quotient rule	$\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = \frac{L}{M},\ M \neq 0$
Power rule	$\lim_{x\to\pm\infty} (f(x))^{r/s} = L^{r/s}$, if r and s are integers and $s \neq 0$
Limit of $\frac{c}{x^r}$	$\lim_{x\to\pm\infty} \frac{c}{x^r} = 0$

squeeze theorem

Conditions	Conclusion
$g(x) \leq f(x) \leq h(x)$ for $x \neq c$	$\lim_{x\to c} f(x) = L$
$\lim_{x\to c} g(x) = L$ and $\lim_{x\to c} h(x) = L$	

Intermediate Value Theorem (IVT)

If a function f is continuous on the interval $[a, b]$ and k is a number between $f(a)$ and $f(b)$, then there is at least one x-value c between a and b such that $f(c) = k$.

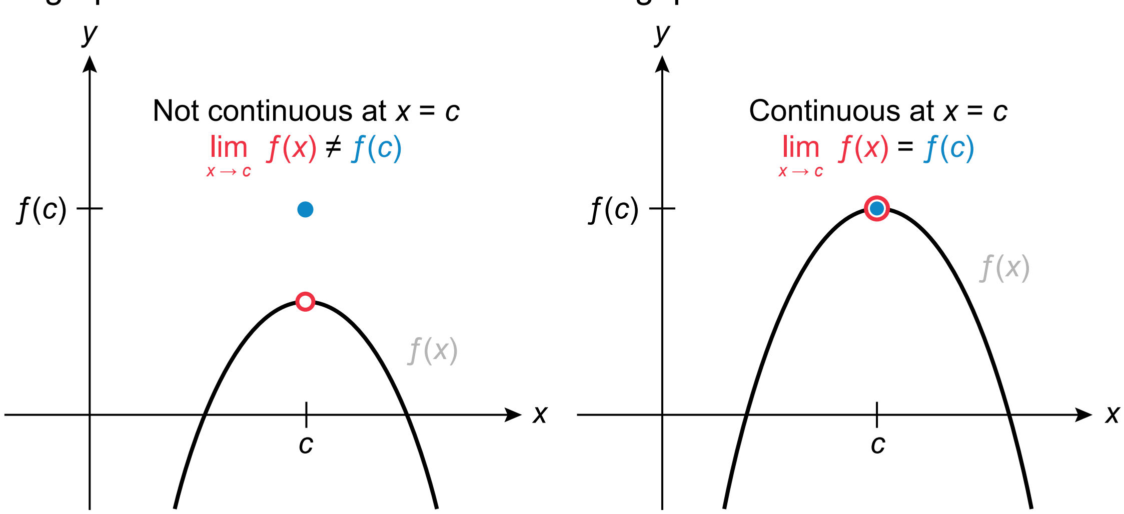


Any continuous function connecting $(a, f(a))$ and $(b, f(b))$ must pass through every y-value between $f(a)$ and $f(b)$ at least once.

A function $f(x)$ is **continuous at $x = c$** if *all* of the following conditions are met:

- $f(c)$ is defined
- $\lim_{x \to c} f(x)$ exists
- $\lim_{x \to c} f(x) = f(c)$

The graph of a continuous function has no "gaps."



Section 2: Derivatives

average rate of change over [*a*, *b*]

$$\frac{f(b) - f(a)}{b - a}$$

definition of derivative of *f* at *x* = *a*

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

definition of derivative of *f* at *x* = *a*

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

f* is differentiable at *x* = *c

$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists and is equal to $f'(c)$

difference quotient

Differentiation rules

Constant	$\frac{d}{dx}[c] = 0$
Power	$\frac{d}{dx}[x^n] = nx^{n-1}$
Natural exponential	$\frac{d}{dx}[e^x] = e^x$
Exponential	$\frac{d}{dx}[a^x] = (\ln a)a^x$
Natural log	$\frac{d}{dx}[\ln(x)] = \frac{1}{x}$
Constant multiple	$\frac{d}{dx}[cf(x)] = cf'(x)$
Sum and difference	$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$

Derivatives of trigonometric functions

$$\frac{d}{dx}[\sin x] = \cos x$$

$$\frac{d}{dx}[\cos x] = -\sin x$$

$$\frac{d}{dx}[\tan x] = \sec^2 x$$

$$\frac{d}{dx}[\cot x] = -\csc^2 x$$

$$\frac{d}{dx}[\csc x] = -\csc(x)\cot(x)$$

$$\frac{d}{dx}[\sec x] = \sec(x)\tan(x)$$

product rule

$$\frac{d}{dx}[uv] = uv' + vu'$$

quotient rule

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$$

chain rule

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

derivative of inverse

$$(f^{-1})'(a) = \frac{1}{f'(b)}$$

Where (b, a) is a point on the graph of f, or a = f(b)

position	$x(t)$	
velocity	$v(t) = x'(t)$	differentiate
acceleration	$a(t) = x''(t)$	

Use **L'Hospital's Rule** to find the limit of the ratio of two differentiable functions $\frac{f(x)}{g(x)}$ as x approaches c. If direct substitution produces one of the indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then differentiate the numerator f and denominator g independently.

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

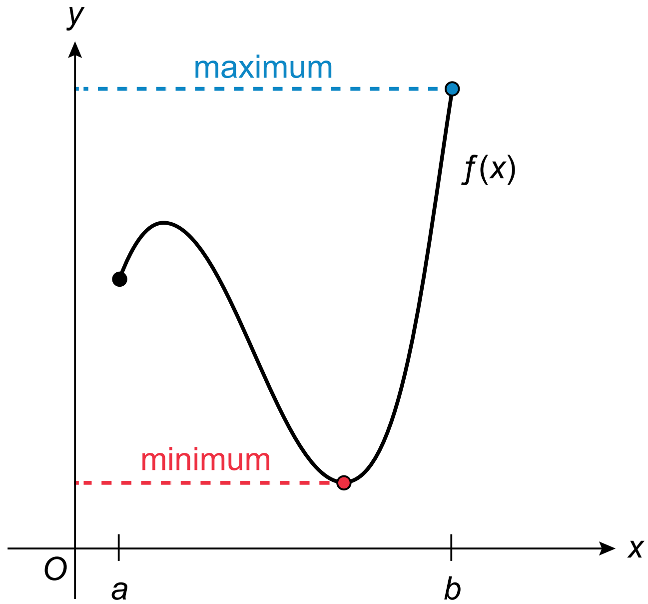
L'Hospital's Rule also applies to limits as $x \to \infty$ or $x \to -\infty$.

Mean Value Theorem

Conditions	Conclusion
f is continuous on $[a, b]$	For some c in (a, b): $f'(c) = \frac{f(b) - f(a)}{b - a}$ instantaneous rate of change at $x = c$; average rate of change on $[a, b]$
f is differentiable on (a, b)	

Extreme Value Theorem (EVT)

If a function f is continuous on the closed interval $[a,b]$, then f is guaranteed to attain an absolute minimum and absolute maximum value on $[a,b]$.



local extrema at $x = c$

$f'(c) = 0$ or undefined

local maximum: $f'(x)$ changes from positive to negative

local minimum: $f'(x)$ changes from negative to positive

second derivative test for critical point at $x = c$

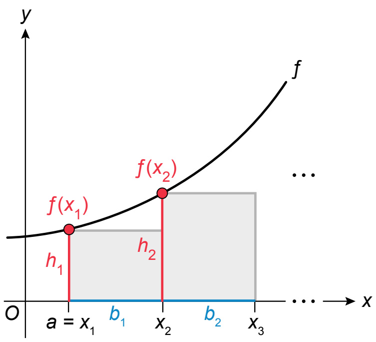
If $f''(c) < 0$	If $f''(c) = 0$	If $f''(c) > 0$
$f(c)$ is a relative maximum	test is inconclusive	$f(c)$ is a relative minimum

Section 3: Integrals and Differential Equations

A **left Riemann sum** approximates the value of a definite integral $\int_a^b f(x)\,dx$. The interval $[a, b]$ is divided into subintervals, and the area bounded by the graph of f and the x-axis on each subinterval is estimated with a rectangle.

The base length b_n of each rectangle is the distance between the endpoints of the subinterval, and the height h_n is the function value at the left endpoint.

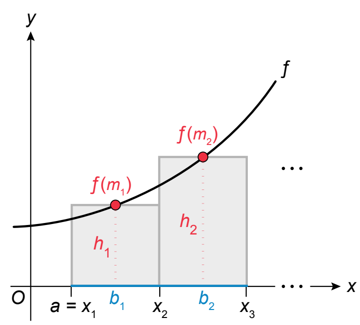
$$\int_a^b f(x)\,dx \approx \underbrace{b_1 h_1}_{\text{area 1st rectangle}} + \underbrace{b_2 h_2}_{\text{area 2nd rectangle}} + \cdots$$



A **midpoint Riemann sum** approximates the value of a definite integral $\int_a^b f(x)\,dx$. The interval $[a, b]$ is divided into subintervals, and the area bounded by the graph of f and the x-axis on each subinterval is estimated with a rectangle.

The base length b_n of each rectangle is the distance between the endpoints of the subinterval, and the height h_n is the function value at the midpoint of the subinterval m_n.

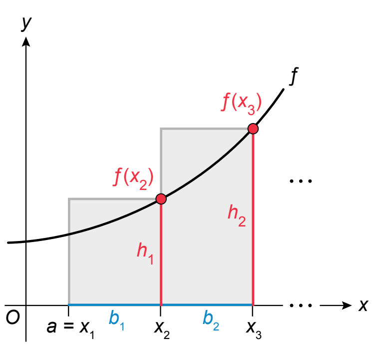
$$\int_a^b f(x)\,dx \approx \underbrace{b_1 h_1}_{\text{area 1st rectangle}} + \underbrace{b_2 h_2}_{\text{area 2nd rectangle}} + \cdots$$



A **right Riemann sum** approximates the value of a definite integral $\int_a^b f(x)\,dx$. The interval $[a, b]$ is divided into subintervals, and the area bounded by the graph of f and the x-axis on each subinterval is estimated with a rectangle.

The base length b_n of each rectangle is the distance between the endpoints of the subinterval, and the height h_n is the function value at the right endpoint.

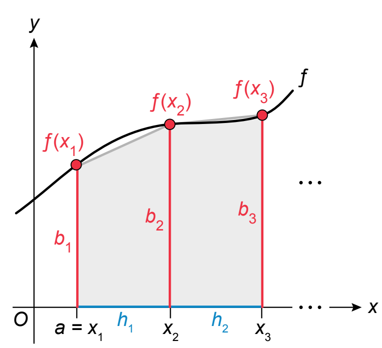
$$\int_a^b f(x)\,dx \approx \underbrace{b_1 h_1}_{\text{area 1st rectangle}} + \underbrace{b_2 h_2}_{\text{area 2nd rectangle}} + \cdots$$



A **trapezoidal sum** approximates the value of a definite integral $\int_a^b f(x)\,dx$. The interval $[a, b]$ is divided into subintervals, and the area bounded by the graph of f and the x-axis on each subinterval is estimated with a trapezoid.

The height h_n of each trapezoid is the distance between the endpoints of the subinterval, and the bases b_n and b_{n+1} are the function values at the endpoints.

$$\int_a^b f(x)\,dx \approx \underbrace{\tfrac{1}{2} h_1(b_1 + b_2)}_{\text{area 1st trapezoid}} + \underbrace{\tfrac{1}{2} h_2(b_2 + b_3)}_{\text{area 2nd trapezoid}} + \cdots$$



Fundamental Theorem of Calculus

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

limit of right Riemann sum

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(a + \Delta x i)\Delta x = \int_a^b f(x)\,dx$$

$a(t)$ acceleration

integrate

$v(t)$ velocity

integrate

$s(t)$ position

Second FTC

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

Fundamental Theorem of Calculus (alternate form)

$$\underbrace{f(b)}_{\text{final quantity}} = \underbrace{f(a)}_{\text{initial quantity}} + \underbrace{\int_a^b f'(t)\,dt}_{\text{net change}}$$

Integration rules

Constant	$\int c\,dx = cx + C$
Power	$\int x^n dx = \frac{x^{n+1}}{n+1} + C$
Constant multiple	$\int cf(x)\,dx = c\int f(x)\,dx$
Sum and difference	$\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
Natural exponential	$\int e^x dx = e^x + C$
Natural log	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$

The following are properties of definite integrals, where functions f and g are continuous on the closed interval $[a, b]$ and a, b, and k are constants.

$$\int_a^a f(x)\,dx = 0$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$

$$\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

improper integral

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

integration by parts

$$\int u\,dv = uv - \int v\,du$$

Integrals of trigonometric functions

$$\int \sin u\,du = -\cos u + C$$

$$\int \cos u\,du = \sin u + C$$

$$\int \sec^2 u\,du = \tan u + C$$

$$\int \csc^2 u\,du = -\cot u + C$$

$$\int (\sec u \tan u)\,du = \sec u + C$$

$$\int (\csc u \cot u)\,du = -\csc u + C$$

$$\int \tan u\,du = -\ln|\cos u| + C$$

$$\int \cot u\,du = \ln|\sin u| + C$$

$$\int \sec u\,du = \ln|\sec u + \tan u| + C$$

$$\int \csc u\,du = -\ln|\csc u + \cot u| + C$$

logistic differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{a}\right)$$

logistic differential equation

$$\frac{dP}{dt} = kP(a - P)$$

Euler's method equation

$$\underbrace{y_{n+1}}_{\text{next } y\text{-value}} = y_n + \underbrace{f'(x_n)}_{\text{derivative at current } x_n\text{-value}} \underbrace{(\Delta x)}_{\text{step size}}$$

Differential equation

constant of proportionality

$$\frac{dy}{dt} = k \cdot y$$

rate of change of y

General solution

constant of proportionality

$$y = C \cdot e^{k \cdot t}$$

time

initial value of y
(value of y when $t = 0$)

average value

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

total distance traveled

$$\int_{t_1}^{t_2} |v(t)|\,dt$$

length of a curve (arc length)

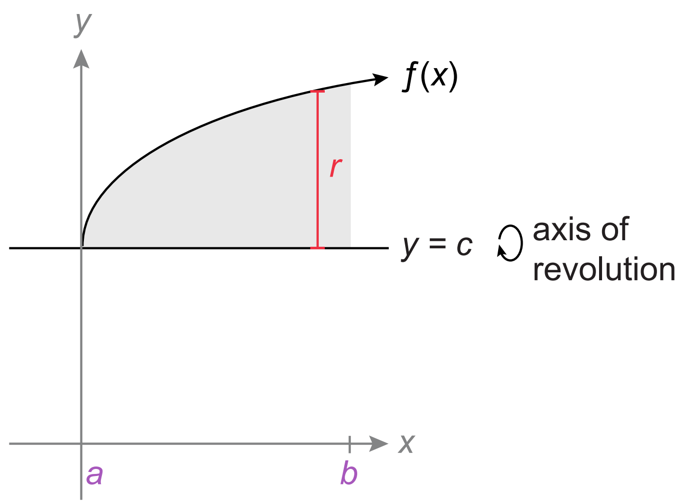
$$\int_a^b \sqrt{1 + (f'(x))^2}\,dx$$

area bounded by two functions on $[x_1, x_2]$

$$A = \int_{x_1}^{x_2} (\text{top} - \text{bottom})\,dx$$

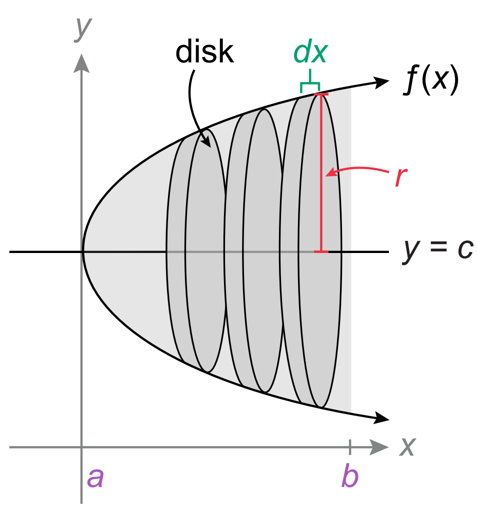
Use the **disk method** to determine the volume of a solid of revolution formed by rotating a region about a horizontal line $y = c$ (axis of revolution) over the interval $a < x < b$ when $y = c$ is a boundary of the region—there is no space between the region and $y = c$.

$$\pi\int_a^b r^2\,dx$$



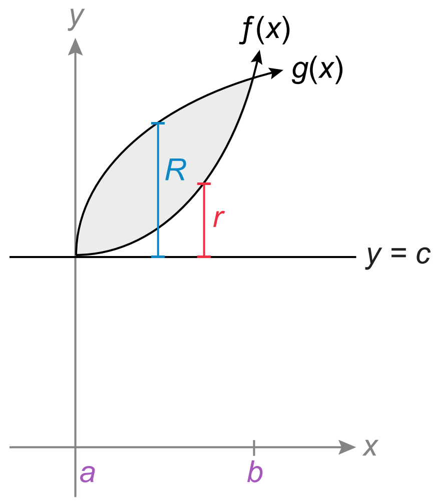
When a region is revolved about an axis of revolution, a perpendicular cross section of the solid is a disk where:

- r is the distance from the axis of revolution to the closest function $f(x)$
- dx is the thickness of the disk



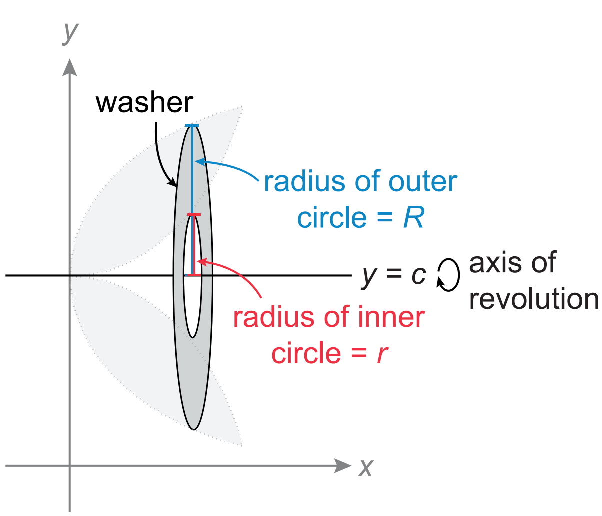
Use the **washer method** to determine the volume of a solid of revolution formed by rotating a region bounded by $f(x)$ and $g(x)$ about a horizontal line $y = c$ (axis of revolution) over the interval $a < x < b$ when $y = c$ is not a boundary of the region—there is space between the region and $y = c$.

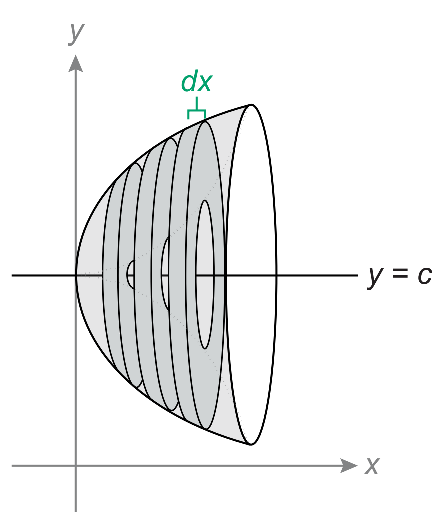
$$\pi\int_a^b ((R(x))^2 - (r(x))^2)dx$$



When a region is revolved about an axis of revolution, a perpendicular cross section of the resulting solid is a disk with a hole (washer) where:

- R is the distance from the axis of revolution to the farthest function $f(x)$
- r is the distance from the axis of revolution to the closest function $g(x)$
- dx is the thickness of the washer





Section 4: Polar Coordinates, Parametric, Equations, and Vector-Valued Functions

derivative of parametric equations

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$$

parametric speed

$$s(t) = \sqrt{(x'(t))^2 + (y'(t))^2}$$

length of parametric curve

$$\int_{t_1}^{t_2} \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\, dt$$

vector differentiation

$$f(t) = \langle x(t), y(t) \rangle$$

$$f'(t) = \langle x'(t), y'(t) \rangle$$

$$f''(t) = \langle x''(t), y''(t) \rangle$$

vector total distance

$$\int_{t_1}^{t_2} \underbrace{\sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}}_{\text{speed}}\, dt$$

slope of a polar curve

$$\frac{dy}{dx} = \frac{\frac{d}{d\theta}[y]}{\frac{d}{d\theta}[x]}$$

polar to rectangular

$$x = r\cos\theta$$

$$y = r\sin\theta$$

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Section 5: Infinite Series

The **harmonic series** is an infinite series given by

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

The harmonic series **diverges** by the *p*-series test.

p-series test

- *p*-series of the form $\frac{1}{n^p}$ converges for $p > 1$

$$\sum_{n=1}^{\infty} \frac{1}{n} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^1},\ p = 1$$

***n*th term test**

$$\sum_{n=1}^{\infty} a_n$$

diverges

$$\lim_{n \to \infty} a_n \neq 0$$

inconclusive

$$\lim_{n \to \infty} a_n = 0$$

A series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$ is called a ***p*-series**.

p-series converges if $p > 1$
p-series diverges if $0 < p \leq 1$

If $p = 1$, the resulting series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ is called a **harmonic series**, which diverges.

sum of geometric series

$$S = \frac{a_1}{1 - r}$$

geometric series

$$\sum_{n=0}^{\infty} ar^n$$

$|r| < 1$

The series converges to $S = \frac{a_1}{1 - r}$, where a_1 is the first term of the series.

$|r| \geq 1$

The series diverges.

integral test

If f is continuous, positive, and eventually decreases as $x \to \infty$, and $\int_c^{\infty} f(x)$

converges

then, $\sum_{n=c}^{\infty} f(n)$ converges

and $\sum_{n=c}^{\infty} f(n) > \int_c^{\infty} f(x)\,dx$

diverges

then, $\sum_{n=c}^{\infty} f(n)$ diverges

alternating series test

$\sum_{n=1}^{\infty} (-1)^n a_n$ converges if:

- $\lim_{n \to \infty} a_n = 0$ and
- a_n is a positive, decreasing sequence

*n*th-degree Taylor polynomial of *f* about *x* = *c*

$$P_n(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x - c)^n$$

Maclaurin polynomial

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \frac{f^{(n)}(0)}{n!}x^n$$

Known power series

$$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots$$